Real Analysis

24·Nov·24

$\boxed{L1.1}$

Start from: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z},\ b \in \mathbb{N} \right\}$

$\mathbb{Q}$	$\mathbb{R}$ (ราณ่ใช้)
$+, -, \times, \div$	$+, -, \times, \div$
$=, >, <, \leq \dots$	$= \quad > \quad < \quad \geq$

$\mathbb{R} = \mathbb{Q}$ add $\underbrace{\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt[3]{5}, \pi, e, \ln 2\}}$

completeness property.

Definition:

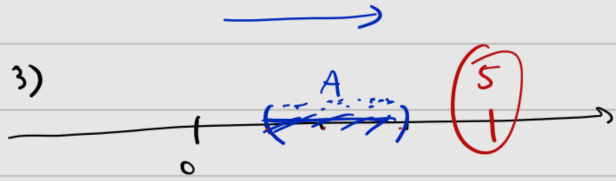
สมบูรณ์ใจของ $\mathbb{R}$ ที่ใช้ $\mathbb{Q} \subset \mathbb{R}$ เพี้ย

$\mathbb{R}$ ของกฎณตัวเลขมูลเรียกว่า "completeness property"

What is completeness property?

1. Bounded Set

Ex: $A = (1, 3)$



Def: • ឧបមា $A \subset \mathbb{R}$. គេថា $M \in \mathbb{R}$ ជាគោលលើរបស់ A កាលណា

$$\forall a \in A, \quad a \leq M.$$

• សំណុំ ដែលមានគោលលើ គេហៅថាសំណុំ ទាល់លើ ។

$\Leftrightarrow$ $A \subseteq \mathbb{R}$ ទាល់លើកាលណា $\exists M \in \mathbb{R}$ ដែល M ជាគោលលើរបស់ A

$\Leftrightarrow$ $A \subseteq \mathbb{R}$ ទាល់លើកាលណា $\exists M \in \mathbb{R}, \ \forall a \in A, \ a \leq M.$

(Remark: តាងផ្នែកគត់ $\lfloor \cdot \rfloor$ ឧ. $\lfloor 1.5 \rfloor = 1$, $\lfloor 3.14 \rfloor = 3$)

Ex:

1). $A = \{1, 5, 7, 8\}$. តើ A ទាល់លើ ឬទេ?

A ពិតជាសំណុំទាល់លើមែន ព្រោះ: បើ $M = 9$. នោះ យើងឃើញថា គ្រប់ធាតុរបស់ A, $(1 \leq 9;\ 5 \leq 9;\ 7 \leq 9;\ 8 \leq 9)$ សុទ្ធតែតូចជាង M.

2) ឧបមា $A = \{1, 5, 7, 8\}$ និង $M = 5.5$. តើ M ជាគោលលើ ?

$1 \leq 5.5$ ✓

$5 \leq 5.5$ ✓

$7 \leq 5.5$ ✗

Answer: M is not an upper bound of A because

គេមាន $a = 7 \in A$ ដែល $a > M$.

3. $A = \mathbb{N} = \{1, 2, 3, 4, \dots\}$

- តើ $M = 100.5$ ជាគោលលើរបស់ A ឬទេ?

No. ព្រោះតែ $120 \in \mathbb{N}$ ដែល

$120 > 100.5$.

$1 \leq 100.5$
$2 \leq 100.5$
$3 \leq 100.5$
$4 \leq 100.5$
$\vdots$
$101 \leq 100.5$ ✗
$102 \leq 100.5$ ✗

- $M = 2025$ is not an upper bound

$\exists\, a = 2030$ ដែល $a > 2025$.

- $M \in \mathbb{R}$. តើ M ជាគោលលើរបស់ A ទេ?

$1 \leq M$ ✓
$2 \leq M$ ✓
$3 \leq M$ ✓
$4 \leq M$ ✓
$5 \leq M$
$6 \leq M$
$\vdots$
$? \leq M$ ✗

M	response $a \in A$
100.5	$a = 101$
2051.891	$a = 2052$
M	$a = \lfloor M \rfloor + 2$ [illegible]

[illegible]: $M \geq 0$

$\forall M$ ที่ใช้ได้ M is not upper bound of A.

$A = \{1, 2, 3, \ldots\}$

Example : $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is not bounded above.

proof: ให้ $M \in \mathbb{R}$ ใดๆ. we will prove that M is not an upper bound.

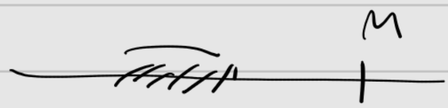
- กรณี $M \le 0$: เลือก $a = 1 \in \mathbb{N}$ ดังนั้น $a > M$
- กรณี $M > 0$: เลือก $a = \lfloor M \rfloor + 1 \in A$

ดังนั้น $a > M$.

In other words, $\forall M \in \mathbb{R}$, M not upper bound of $\mathbb{N}$

Therefore $\boxed{\mathbb{N} \text{ is not bounded above}}$. ▨

ให้ $A \subseteq \mathbb{R}$, $M \in \mathbb{R}$



- M เป็นขอบเขตบน ก็ต่อเมื่อ $\forall a \in A$, $a \le M$.

  M ไม่เป็นขอบเขตบน ก็ต่อเมื่อ $\exists a \in A$, $a > M$.

- A is bounded above ก็ต่อเมื่อ $\exists M \in \mathbb{R}, \forall a \in A$, $a \le M$

  A is not bounded above ก็ต่อเมื่อ: $\forall M \in \mathbb{R}$, $\exists a \in A$, $a > M$.

Example : $A = \{1, 3, 5, 7, 9, 11, 13, 15 \ldots\} = \{2n-1 : n \in \mathbb{N}\}$

ເຮົາພິສູດວ່າ A ບໍ່ມີຂອບເທິງ .

Sketch : $\forall M \in \mathbb{R}$, M not upper bound

- $M = 24$ not upper bound of A. ເຊັ່ນ: $25 \in A$, $25 > M$
- $M = 73$ —— " —— ເຊັ່ນ: $75 \in A$, $75 > M$
- $M = 312.7$ —— " —— ເຊັ່ນ: $313 \in A$, $313 > M$
- $M < 0$ —— " —— ເຊັ່ນ: $1 \in A$, $1 > M$
- $M > 0$ ຫຼາຍໆ ເພື່ອ a ເລືອກ = ? so that $a > M$
  - $a = 2M+1$ still not work, cuz a might not ຄີກ
  - update $a = 2\lfloor M \rfloor + 1 \in A$ ✓✓

Proof : ເຮົາພິສູດວ່າ A ບໍ່ມີຂອບເທິງ

ໃຫ້ $M \in \mathbb{R}$.

- ກໍລະນີ $M \leq 0$: ເອົາ $a = 1 \in A$, $a > M$
- ກໍລະນີ $M > 0$: ເອົາ $a = 2\lfloor M \rfloor + 1 \in A$ ເຊິ່ງ $a > M$.

ດັ່ງນັ້ນ: M ບໍ່ແມ່ນຂອບເທິງຂອງ A

ສະຫຼຸບ: $\boxed{\text{A ບໍ່ມີຂອບເທິງ}}$ ∎

Recall :

សំណុំចំនួនពិត ✓

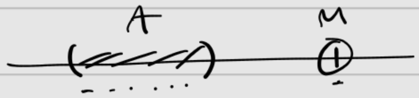
$\mathbb{Q}$	$\mathbb{R}$
$+, \times, -, \div$	$+, \cdots$
$\leq$	$\leq$
doesn't have	completeness property

$\mathbb{R} = \mathbb{Q} +$ completeness

- ស្គាល់ $\mathbb{Q}$
- completeness.   ~~$\sqrt{2}, \sqrt{3}, \pi, \ln 2$~~
- $\lfloor 2.5 \rfloor = 2$

  $\lfloor 3.14 \rfloor = 3$

  $\lfloor -3.14 \rfloor = -4$



Def: [upper bound] : $A \subseteq \mathbb{R}$. គេថា $M \in \mathbb{R}$ ជាគោលលើ $\forall a, a \leq M$

Def: $A \subseteq \mathbb{R}$ ទាល់លើកាលណា $\exists M \in \mathbb{R}, \forall a \in A, a \leq M$.

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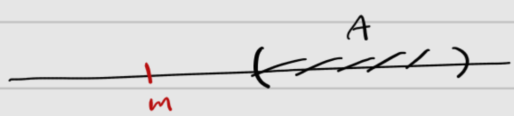
Def: [lower bound] យក $m \in \mathbb{R}$ ហើយ $A \subseteq \mathbb{R}$.

គេថា m ជាគោលក្រោមរបស់ A កាលណា $\forall a \in A, a \geq m$.



Def: [គេថា $A \subseteq \mathbb{R}$ សំណុំទាល់ក្រោម កាលណា $\exists m \in \mathbb{R}, \forall a \in A, a \geq m$.]

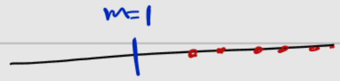
Example: bounded below / bounded above ?

(1): $A = \{ \frac{n}{2}+1 : n \in \mathbb{N} \}$

$= \{ \frac{1}{2}+1, \frac{2}{2}+1, \frac{3}{2}+1, \frac{4}{2}+1, \dots \}$

- bounded below.

ເຫັນໄດ້ງ່າຍວ່າ $m=1$ ແມ່ນຂອບເຂດລຸ່ມ.



ທຸກທາດຂອງເຊດ $\frac{n}{2}+1$, $n \in \mathbb{N}$ ເພາະ

$$\frac{n}{2}+1 > m \quad \forall n \in \mathbb{N}$$

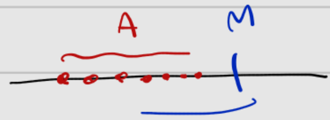
$\Rightarrow$ m ເປັນຂອບເຂດລຸ່ມຂອງ A

$\Rightarrow$ A bounded below.

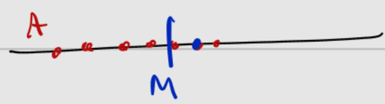
(Def : A ມີຂອບເຂດເທິງ $\rightarrow$ $\exists M$ M ຂອບເຂດເທິງ
A ບໍ່ມີຂອບເຂດເທິງ $\rightarrow$ $\forall M \in \mathbb{R}$, M ບໍ່ແມ່ນຂອບເຂດເທິງ)

(2) $A = \{ \frac{n}{2}+1 : n \in \mathbb{N} \}$ ບໍ່ມີຂອບເຂດເທິງ

__ພິສູດ__ : ຕ້ອງພິສູດວ່າ $\forall M \in \mathbb{R}$, M ບໍ່ແມ່ນຂອບເຂດເທິງຂອງ A.



M ແມ່ນຂອບເຂດເທິງ : $\forall a \in A$, $a \leq M$



M ບໍ່ແມ່ນຂອບເຂດເທິງ : $\exists a \in A$, $a > M$.

$A = \{ \frac{n}{2}+1 : n \in \mathbb{N} \} = \{ \frac{1}{2}+1, \frac{2}{2}+1, \frac{3}{2}+1, \frac{4}{2}+1, \dots \}$

$M = 5$	$\exists a = \frac{10}{2}+1 = 6$, $a > M$
$M = 10.5$	$\exists a = \frac{20}{2}+1 \in A$, $a > M$

$M = 1.7$	$a = \frac{2\lfloor M \rfloor}{2} + 1 \in A$ ✓
	$\Rightarrow a = \lfloor M \rfloor + 1 > M$ ✓
$M = -20$	$a = \frac{1}{2} + 1 \in A$, $a > M$

$A = \{ \frac{n}{2} + 1 : n \in \mathbb{N} \}$

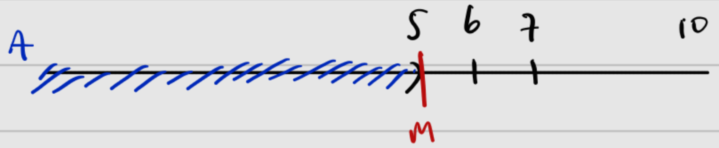
ស្រាយ: we need to prove that $\forall M \in \mathbb{R}$, M is not upper bound.

- if $M \leq 0$: choose $a = \frac{1}{2} + 1 \in A$, $a > M$
- if $M > 0$: choose $a = \frac{2\lfloor M \rfloor}{2} + 1 \in A$ and

$$a = \lfloor M \rfloor + 1 > M$$

Therefore the set A is not bounded above.

Example: $A = (-\infty, 5) = \{ a : a < 5 \}$ ទល់លើ



$M = 6$ គោលលើ
$M = 5.5$ គោលលើ
$M = 5$: គោលលើ

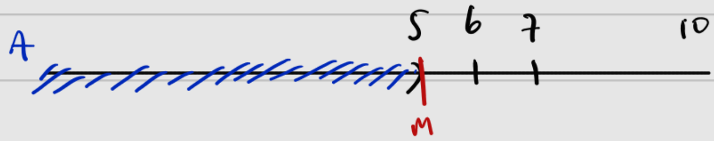
គោលលើដែលតូចជាងគេ "best"
we call it supremum of A
$5 = \sup A$

~~$M = 4.9999$~~: ✗ not upper bound

~~$M = 4.9$~~ not upperbound : $\exists a \in 4.99$, $a > M$

to talk about "least upper bound" of A

- A bound above



Def: [supremum) Let $A \subseteq \mathbb{R}$ be bounded above.

We say that $M = \sup A$ if

(i) M is an upper bound

(ii) $\forall \varepsilon > 0$, $M - \varepsilon$ not upper bound of A

សមមូលគ្នា

Def: [supremum) Let $A \subseteq \mathbb{R}$ be bounded above.

We say that $M = \sup A$ if

(i) M is an upper bound

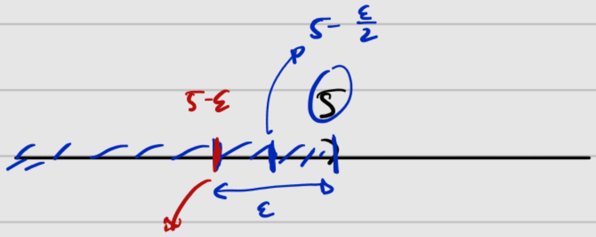
(ii) $\forall \varepsilon > 0$, $\exists a \in A$, $a > M - \varepsilon$.

Ex: $A = (-\infty, 5] = \{a \in \mathbb{R} : a \leq 5\}$ (ស្រាយថា $\sup A = 5$.

Sketch:

(i) 5 ជាគោលលើ ✓

(ii) យក $\varepsilon > 0$ ដូចម្ដេចក៏ដោយ (ត្រូវស្រាយថា $5-\varepsilon$ មិនមែនជាគោលលើ



not upper bound ?? $\Leftrightarrow$ រក $a \in A$ ដែល $a > 5-\varepsilon$

$(a > 5-\varepsilon)$

យក $a = 5 - \varepsilon/2$. ដោយ $a < 5$ នាំឲ្យ: $a \in A$ ហើយ

$$a - (5-\varepsilon) = 5 - \varepsilon/2 - 5 + \varepsilon/2 = \varepsilon/2 > 0$$

$\Rightarrow$ $a > 5 - \varepsilon$

$\Rightarrow$ $5-\varepsilon$ មិនមែនជាគោលលើ $\forall \varepsilon > 0$.

ដូចនេះ: $\boxed{5 = \sup A}$