TD n°0 – Warm up exercises

Exercise 1. Prove that $|x| = \max\{x, -x\}$.

Exercise 2 (Maximum and minimum)**.** Let x, y be real numbers. Prove that

$$\max\{x, y\} = \frac{x + y + |x - y|}{2},$$

and find a similar formular for $\min\{x, y\}$.

Exercise 3. Let $\varepsilon > 0$ and $a, x \in \mathbb{R}$. Prove that $|x - a| < \varepsilon \iff a - \varepsilon < x < a + \varepsilon$.

Exercise 4 (Triangle inequality)**.** Let x and y be real numbers.

(1). Prove that $|x + y| \leq |x| + |y|$.

(2). Prove that $|x - y| \geq ||x| - |y||$.

(3). Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

Exercise 5 (Cauchy inequality)**.**

(1). For all $a, b \geq 0$, prove that $a + b \geq 2\sqrt{ab}$.

(2). (Generalized Cauchy inequality) Let $a_1, a_2, \ldots, a_n$ be *positive* real numbers. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Exercise 6 (Cauchy–Schwarz inequality)**.** Let $n \in \mathbb{N}$ and let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Prove that

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

Exercise 7 (Bernoulli inequality)**.** Let $n \in \mathbb{N}$. Prove that if $a \geq -1$ then

$$(1 + a)^n \geq 1 + na.$$

Exercise 8. Let $\varepsilon > 0$ and $x_0, y_0 \in \mathbb{R}$. Suppose that x, y satisfy

$$|x - x_0| \leq \min\left\{1, \ \frac{\varepsilon}{2(1 + |y_0|)}\right\} \quad \text{and} \quad |y - y_0| \leq \frac{\varepsilon}{2(1 + |x_0|)}.$$

Prove that $|xy - x_0 y_0| \leq \varepsilon$.